

HÖLDER CONTINUITY FOR STOCHASTIC FRACTIONAL HEAT EQUATION WITH COLORED NOISE

KEXUE LI

ABSTRACT. In this paper, we consider semilinear stochastic fractional heat equation $\frac{\partial}{\partial t}u_{\beta,t}(x) = \Delta^{\alpha/2}u_{\beta,t}(x) + \sigma(u_{\beta,t}(x))\eta_{\beta}$. The Gaussian noise η_{β} is assumed to be colored in space with covariance of the form $E(\eta_{\beta}(t, x)\eta_{\beta}(s, y)) = \delta(t - s)f_{\beta}(x - y)$, where f_{β} is the Riesz kernel $f_{\beta}(x) \propto |x|^{-\beta}$. We obtain the spatial and temporal Hölder continuity of the mild solution.

1. INTRODUCTION

In this paper, we consider the following stochastic fractional heat equation

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t}u_{\beta,t}(x) = \Delta^{\alpha/2}u_{\beta,t}(x) + \sigma(u_{\beta,t}(x))\eta_{\beta} & , \quad t > 0, \quad x \in R, \\ u_{\beta,0}(x) = \phi(x), \end{cases}$$

where $1 < \alpha \leq 2$, $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian defined by the Fourier transform

$$(\mathcal{F}(-\Delta)^{\alpha/2}u)(\xi) = (2\pi|\xi|)^{\alpha}\mathcal{F}(u)(\xi),$$

here \mathcal{F} denotes the Fourier transform,

$$(1.2) \quad (\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} \varphi(x) dx.$$

η_{β} is the Gaussian space time colored noise with covariance of the form

$$(1.3) \quad E[\eta_{\beta}(t, x)\eta_{\beta}(s, y)] = \delta(t - s)f_{\beta}(x - y),$$

where ([9], Ex.1)

$$(1.4) \quad f_{\beta}(x) = c_{1-\beta}g_{\beta}(x) = (\mathcal{F}g_{1-\beta})(x), \quad g_{\beta}(x) = \frac{1}{|x|^{\beta}}, \quad \beta \in (0, 1),$$

and

$$(1.5) \quad c_{\beta} = \frac{2 \sin(\beta\pi/2)\Gamma(1 - \beta)}{(2\pi)^{1-\beta}},$$

where $\Gamma(\cdot)$ is the Gamma function.

We assume that the following conditions hold:

(A1) ϕ is bounded and ρ -Hölder continuous.

(A2) σ is Lipschitz continuous and there exists a constant K such that $|\sigma(x) - \sigma(y)| \leq K|x - y|$ and $|\sigma(x)| \leq K(1 + |x|)$.

2010 *Mathematics Subject Classification.* 35K55.

Key words and phrases. Stochastic fractional heat equation; fractional heat kernel; colored noise; Hölder continuity.

The mild solutions are the solutions of the integral equations

$$(1.6) \quad u_{\beta,t}(y) = (u_{\beta,0} * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u_{\beta,s}(x)) \eta_{\beta}(ds, dx).$$

where the fractional heat kernel $p_t(x)$ is the fundamental solution of

$$(1.7) \quad v_t = \Delta^{\alpha/2} v,$$

and $*$ denotes the usual convolution operator, $(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$. Since $0 < \beta < 1$, we can get the existence and uniqueness of the mild solution of (1.1) (see, e.g., [11, 12]). It is known that $p_t(x)$ satisfies the following inequality ([6, 5, 7])

$$(1.8) \quad \frac{c_1 t}{(t^{1/\alpha} + |x|)^{1+\alpha}} \leq p_t(x) \leq \frac{c_2 t}{(t^{1/\alpha} + |x|)^{1+\alpha}},$$

where $t > 0$, $x \in \mathbb{R}$, c_1 and c_2 are positive constants depending on α .

In the very recent paper [1], Bezdek considered the following equations

$$(1.9) \quad \begin{cases} \frac{\partial}{\partial t} u_{\beta,t}(x) = \frac{\kappa}{2} \Delta u_{\beta,t}(x) + \sigma(u_{\beta,t}(x)) \eta_{\beta} & , \quad t > 0, \quad x \in \mathbb{R}, \\ u_{\beta,0}(x) = \phi(x), \end{cases}$$

where $\kappa > 0$ and η_{β} is the Gaussian noise colored in space and white in time. Stochastic PDEs with colored noise has been studied in many papers (see, e.g., [3, 4, 8, 9]).

Bezdek has obtained the Hölder continuity estimates which take into account β as a variable, the results are novel in that sense. In this paper, based on some estimates of the fractional heat kernel, we will show the spatial and temporal Hölder continuity for the mild solution of stochastic fractional heat equations (1.1).

2. THE SPATIAL AND TEMPORAL HÖLDER CONTINUITY

2.1. Some lemma. In this subsection, we will prove some lemmas, which will be used in next subsections. We use C to denote generic constant, which may change from line to line.

Lemma 2.1. *For all $t > 0$ and $x \in \mathbb{R}$,*

$$(2.1) \quad \int_{\mathbb{R}} |p_t(y-x) - p_t(y)| dy \leq C \left(\frac{|x|}{t^{1/\alpha}} \wedge 1 \right),$$

where C does not depend on t or x .

Proof. For all $r > 0$, define

$$(2.2) \quad \mu(r) = \mu(r, t) := \sup_{z \in \mathbb{R}, |z| \leq r} \int_{\mathbb{R}} |p_t(y-z) - p_t(y)| dy.$$

Then

$$(2.3) \quad \mu(|x|) = \sup_{z \in (0, |x|)} \int_{-\infty}^{\infty} \left| \int_{y-z}^y \frac{\partial p_t(\xi)}{\partial \xi} d\xi \right| dy.$$

By (2.3) of [6] (or Lemma 5 in [7]), we have

$$(2.4) \quad \left| \frac{\partial p_t(\xi)}{\partial \xi} \right| \leq C \frac{t|\xi|}{(t^{1/\alpha} + |\xi|)^{3+\alpha}},$$

where C only depends on α .
Taking (2.4) into (2.28) to get

$$\begin{aligned}
 \mu(|x|) &\leq C|x| \int_{-\infty}^{\infty} \frac{t|w|}{(t^{1/\alpha} + |\xi|)^{3+\alpha}} d\xi \\
 &= \frac{C|x|}{t^{1/\alpha}} \int_{-\infty}^{\infty} \frac{|\nu|}{(1 + |\nu|)^{3+\alpha}} d\nu \\
 (2.5) \quad &\leq \frac{C|x|}{t^{1/\alpha}}.
 \end{aligned}$$

On the other hand, since $|p_t(y-x) - p_t(y)| \leq p_t(y-x) + p_t(y)$ and $\int_{\mathbb{R}} p_t(y) dy = 1$, we have $\mu(|x|) \leq 2$. \square

Lemma 2.2. *For all $t, \varepsilon > 0$, we have*

$$(2.6) \quad \int_{\mathbb{R}} |p_{t+\varepsilon}(y) - p_t(y)| dy \leq C(\log(t+\varepsilon) - \log(t)) \wedge 1.$$

Proof. From (1.7) and Proposition 2.1 in [2], it is easy to show that

$$(2.7) \quad \left| \frac{\partial p_t(y)}{\partial t} \right| \leq \frac{C p_t(y)}{t}.$$

Then we have

$$\begin{aligned}
 \int_{\mathbb{R}} |p_{t+\varepsilon}(y) - p_t(y)| dy &= \int_{\mathbb{R}} \left| \int_t^{t+\varepsilon} \frac{\partial p_s(y)}{\partial s} ds \right| dy \leq C \int_{\mathbb{R}} \int_t^{t+\varepsilon} \frac{p_s(y)}{s} ds dy \\
 (2.8) \quad &= C \int_t^{t+\varepsilon} \frac{1}{s} ds = C(\log(t+\varepsilon) - \log(t)).
 \end{aligned}$$

On the other hand, we have $\int_{\mathbb{R}} |p_{t+\varepsilon}(y) - p_t(y)| dy \leq 2$. \square

Lemma 2.3. *Let $0 < \rho < 1$ and let w be a bounded ρ -Hölder continuous function, then there exists $C > 0$ such that for every $t > 0$, $\delta > 0$, $x \in \mathbb{R}$, $z \in \mathbb{R}$, we have*

$$\begin{aligned}
 \left| \int_{\mathbb{R}} (p_t(x-y) - p_t(z-y)) w(y) dy \right| &\leq C|x-z|^\rho, \\
 \left| \int_{\mathbb{R}} (p_{t+\delta}(x-y) - p_t(x-y)) w(y) dy \right| &\leq C\delta^{\rho/\alpha}.
 \end{aligned}$$

Proof.

$$\begin{aligned}
 \left| \int_{\mathbb{R}} (p_t(x-y) - p_t(z-y)) w(y) dy \right| &= \left| \int_{\mathbb{R}} p_t(z-y) w(y+x-z) dy - \int_{\mathbb{R}} p_t(z-y) w(y) dy \right| \\
 &= \left| \int_{\mathbb{R}} p_t(z-y) (w(y+x-z) - w(y)) dy \right| \leq C|x-z|^\rho \int_{\mathbb{R}} p_t(z-y) dy = C|x-z|^\rho.
 \end{aligned}$$

By the semigroup property of p_t and note that $\int_{\mathbb{R}} p_{\delta}(y)dy = \int_{\mathbb{R}} p_t(x-z)dz = 1$,

$$\begin{aligned}
& \left| \int_{\mathbb{R}} (p_{t+\delta}(x-y) - p_t(x-y))w(y)dy \right| = \left| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} p_t(x-z)p_{\delta}(z-y)dz \right) w(y)dy - \int_{\mathbb{R}} p_t(x-y)w(y)dy \right| \\
& = \left| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} p_t(x-z)p_{\delta}(y)dz \right) w(z-y)dy - \int_{\mathbb{R}} p_t(x-z)w(z)dz \right| \\
& = \left| \int_{\mathbb{R}} p_{\delta}(y) \left(\int_{\mathbb{R}} p_t(x-z)w(z-y)dz \right) dy - \int_{\mathbb{R}} p_{\delta}(y) \left(\int_{\mathbb{R}} p_t(x-z)w(z)dz \right) dy \right| \\
(2.9) \quad & \leq \int_{\mathbb{R}} p_{\delta}(y)|y|^{\rho}dy.
\end{aligned}$$

From (2.9) and (1.8), it follows that

$$\int_{\mathbb{R}} (p_{t+\delta}(x-y) - p_t(x-y))w(y)dy \leq 2 \int_0^{\infty} \frac{\delta y^{\rho}}{(\delta^{1/\alpha} + y)^{1+\alpha}} dy = 2\delta^{\rho/\alpha} \int_0^{\infty} \frac{y^{\rho}}{(1+y)^{1+\alpha}} dy.$$

□

2.2. Difference in the spatial variable. For a random variable $X \in L^k(P)$, define $\|X\|_{L^k(P)} = (E(|X|^k))^{1/k}$. For simplicity, we write $\|\cdot\|_k$ instead of $\|\cdot\|_{L^k(P)}$. We will estimate the spatial and the time difference of the following stochastic integral I in this and next subsection. For $t \in [0, T]$, $x, y, z \in \mathbb{R}$, Define

$$(2.10) \quad I_{\beta, t}(x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(z-x) \sigma(u_{\beta, s}(z)) \eta_{\beta}(ds, dz),$$

and denote

$$\begin{aligned}
A_s(x, y) &= \sigma(u_{\beta, s}(x)) \sigma(u_{\beta, s}(y)), \\
B_s(r) &= p_{t-s}(r-x) - p_{t-s}(r-y).
\end{aligned}$$

For all $k \geq 2$, the difference in the spatial variable is

$$E(|I_{\beta, t}(x) - I_{\beta, t}(y)|^k) = E\left(\left| \int_0^t \int_{\mathbb{R}} (p_{t-s}(z-x) - p_{t-s}(z-y)) \sigma(u_{\beta, s}(z)) \eta_{\beta}(ds, dz) \right|^k\right).$$

Theorem 2.4. For all $t \in [0, T]$, $x, y \in \mathbb{R}$,

$$(2.11) \quad E(|I_{\beta, t}(x) - I_{\beta, t}(y)|^k) \leq C|x-y|^{\frac{\alpha b k}{2}},$$

where C is a constant, $b \in (0, 1 - \frac{1}{\alpha})$, $\alpha \in (1, 2]$, $\beta \in (0, 1)$.

Proof. We apply the Cauchy-Schwarz inequality to bound $E(|\sigma(u_{\beta, s}(x)) \sigma(u_{\beta, s}(y))|^{k/2})$ by $\sup_{x \in \mathbb{R}} E|\sigma(u_{\beta, s}(x))|^k$. Similar to the proof of Theorem 13 of [9], we can show that $\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} E|u_{\beta, s}(x)|^k < \infty$. Then by (A2), we obtain $\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} E|\sigma(u_{\beta, s}(x))|^k < \infty$. It is easy to show that $p_r * f$ is positive definite and continuous for all $r > 0$, then $\sup_{z \in \mathbb{R}} (p_{t-s} * f_{\beta})(z) = (p_{t-s} * f)(0)$. Since $(B_s * f_{\beta})(w) \leq 2 \sup_{z \in \mathbb{R}} (p_{t-s} * f)(z)$, we get $\sup_{z \in \mathbb{R}} (p_{t-s} * f_{\beta})(z) \leq 2(p_{t-s} * f)(0)$. By Burkholder inequality ([13], Theorem

5.27), Minkowski integral inequality ([14], Appendice A.1) and Lemma 2.1 we have

$$\begin{aligned}
& E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \\
& \leq c_k E\left(\left|\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_{\beta}(z-w) B_s(z) B_s(w) A_s(z,w) ds dz dw\right|^{k/2}\right) \\
& \leq c_k \left|\int_0^t \sup_{x \in \mathbb{R}} \|\sigma(u_{\beta,s}(x))\|_k^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f_{\beta}(z-w) |B_s(z)| |B_s(w)| ds dz dw\right|^{k/2} \\
& \leq C \left|\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_{\beta}(z-w) |B_s(z)| |B_s(w)| ds dz dw\right|^{k/2} \\
& \leq C \left|\int_0^t (p_{t-s} * f_{\beta})(0) ds \int_{\mathbb{R}} |p_{t-s}(z-x) - p_{t-s}(y-x)| dz\right|^{k/2} \\
(2.12) \quad & \leq C \left|\int_0^t (p_{t-s} * f_{\beta})(0) \left(\frac{|x-y|}{(t-s)^{1/\alpha}} \wedge 1\right) ds\right|^{k/2}.
\end{aligned}$$

Since $r \wedge 1 \leq r^{\alpha b}$ for all $r > 0$ and $b \in (0, \frac{1}{\alpha})$, by (2.12), we have

$$(2.13) \quad E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \leq C |x-y|^{\frac{\alpha b k}{2}} \left|\int_0^t (p_{t-s} * f_{\beta})(0) (t-s)^{-b} ds\right|^{k/2}.$$

By (1.8), we have

$$\begin{aligned}
& (p_{t-s} * f_{\beta})(0) = c_{1-\beta} \int_{\mathbb{R}} \frac{1}{|x|^{\beta}} p_{t-s}(x) dx \leq c_{1-\beta} \int_{\mathbb{R}} \frac{1}{|x|^{\beta}} \cdot \frac{t-s}{((t-s)^{1/\alpha} + |x|)^{1+\alpha}} dx \\
(2.14) \quad & \leq c_{1-\beta} (t-s)^{-\beta/\alpha} \int_{\mathbb{R}} \frac{1}{|r|^{\beta}(1+|r|)^{1+\alpha}} dr \leq C (t-s)^{-\beta/\alpha}.
\end{aligned}$$

Put (2.14) into (2.13) to get

$$(2.15) \quad E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \leq C |x-y|^{\frac{\alpha b k}{2}} \left|\int_0^t (t-s)^{-\frac{\beta}{\alpha}-b} ds\right|^{k/2}.$$

Since $\beta \in (0, 1)$ and $\alpha \in (1, 2]$, we can choose $b \in (0, 1 - \frac{1}{\alpha}) \subset (0, \frac{1}{\alpha})$ to guarantee that $\left|\int_0^t (t-s)^{-\frac{\beta}{\alpha}-b} ds\right| < \infty$. Therefore we obtain

$$(2.16) \quad E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \leq C |x-y|^{\frac{\alpha b k}{2}}, \quad t \in [0, T], \quad x, y \in \mathbb{R}.$$

□

2.3. Difference in the time variable. For all $k \geq 2$, the difference in the time variable is

$$\begin{aligned}
& E(|I_{\beta,t+\delta} - I_{\beta,t}|^k) \\
& = E\left(\left|\int_0^{t+\delta} \int_{\mathbb{R}} p_{t+\delta-s}(z-x) \sigma(u_{\beta,s}(z)) \eta_{\beta}(ds, dz) - \int_0^t \int_{\mathbb{R}} p_{t-s}(z-x) \sigma(u_{\beta,s}(z)) \eta_{\beta}(ds, dz)\right|^k\right)
\end{aligned}$$

Theorem 2.5. For all $\delta > 0$, $t \in [0, T]$, $x, y \in \mathbb{R}$,

$$(2.17) \quad E(|I_{\beta,t+\delta}(x) - I_{\beta,t}(x)|^k) \leq C \delta^{\frac{(\alpha-\beta)k}{2\alpha}},$$

where C is a constant, $\alpha \in (1, 2]$, $\beta \in (0, \frac{\alpha}{2}]$.

Proof. By the elementary inequality $|a + b|^k \leq 2^k |a|^k + 2^k |b|^k$, we have

$$\begin{aligned}
& E(|I_{\beta,t+\delta}(x) - I_{\beta,t}(x)|^k) \\
& \leq 2^k E(|\int_0^{t+\delta} \int_{\mathbb{R}} (p_{t+\delta-s}(z-x) - p_{t-s}(z-x)) \sigma(u_{\beta,s}(z)) \eta_{\beta}(ds, dz)|^k) \\
& \quad + 2^k E(|\int_t^{t+\delta} \int_{\mathbb{R}} p_{t+\delta-s}(z-x) \sigma(u_{\beta,s}(z)) \eta_{\beta}(ds, dz)|^k) \\
(2.18) \quad & = I_1 + I_2.
\end{aligned}$$

For I_2 , by the same technique as in the proof of Theorem 2.4, we have

$$\begin{aligned}
I_2 & \leq C \left(\int_t^{t+\delta} \sup_{x \in \mathbb{R}} \|\sigma(u_{\beta,s}(x))\|_k^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f_{\beta}(z-w) p_{t+\delta-s}(z-x) p_{t+\delta-s}(w-x) ds dz dw \right)^{k/2} \\
(2.19) \quad & \leq C \left(\int_t^{t+\delta} \int_{\mathbb{R}} \int_{\mathbb{R}} f_{\beta}(z-w) p_{t+\delta-s}(z-x) p_{t+\delta-s}(w-x) ds dz dw \right)^{k/2}.
\end{aligned}$$

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapid decreasing test-functions from \mathbb{R} to \mathbb{R} , by elementary properties of convolution and Fourier transform, the following holds (see formula (10) in [9]):

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) f_{\beta}(x-y) \varphi(y) dx dy = \int_{\mathbb{R}} f_{\beta}(x) (\varphi * \tilde{\psi})(x) dx = \int_{\mathbb{R}} g_{1-\beta}(\xi) |\mathcal{F}\varphi(\xi)|^2 d\xi,$$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$, where $\tilde{\psi}$ is defined by $\tilde{\psi}(x) = \psi(-x)$. Then by change of variables,

$$(2.20) \quad \int_{\mathbb{R}} \int_{\mathbb{R}} p_{t+\delta-s}(z-x) f_{\beta}(z-w) p_{t+\delta-s}(w-x) dz dw = \int_{\mathbb{R}} g_{1-\beta}(\xi) |\mathcal{F}p_{t+\delta-s}(\xi)|^2 d\xi.$$

Recall the Fourier transform of fractional heat kernel $p_t(x)$ (see formula (3) in [7]) and note (1.2), we have

$$(2.21) \quad \mathcal{F}p_{t-s}(\xi) = e^{-(t-s)(2\pi|\xi|)^{\alpha}}.$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}} g_{1-\beta}(\xi) |\mathcal{F}p_{t-s}(\xi)|^2 d\xi & = \int_{\mathbb{R}} g_{1-\beta}(\xi) e^{-(t-s)2^{\alpha+1}\pi^{\alpha}|\xi|^{\alpha}} d\xi \\
& = \int_{\mathbb{R}} |\xi|^{\beta-1} e^{-(t-s)2^{\alpha+1}\pi^{\alpha}|\xi|^{\alpha}} d\xi \\
& = 2 \int_0^{\infty} \xi^{\beta-1} e^{-(t-s)2^{\alpha+1}\pi^{\alpha}\xi^{\alpha}} d\xi \\
& = \frac{2}{\alpha} \int_0^{\infty} e^{-(t-s)2^{\alpha+1}\pi^{\alpha}r} r^{\frac{\beta}{\alpha}-1} dr \\
& = \frac{2}{\alpha(2^{\alpha+1}\pi^{\alpha}(t-s))^{\beta/\alpha}} \int_0^{\infty} e^{-z} z^{\frac{\beta}{\alpha}-1} dz \\
(2.22) \quad & = \frac{2\Gamma(\frac{\beta}{\alpha})}{\alpha(2^{\alpha+1}\pi^{\alpha}(t-s))^{\beta/\alpha}}.
\end{aligned}$$

By (2.19), (2.20) and (2.22), we get

$$(2.23) \quad I_2 \leq C \left(\int_t^{t+\delta} (t+\delta-s)^{-\beta/\alpha} ds \right)^{k/2} = C \delta^{\frac{(\alpha-\beta)k}{2\alpha}}.$$

For I_1 , by the similar argument and note that Lemma 2.2 and (2.14), we have

$$(2.24) \quad \begin{aligned} I_1 &\leq C \left(\int_0^t \sup_{x \in \mathbb{R}} \|\sigma(u_{\beta,s}(x))\|_k^2 (f_\beta * p_{t-s})(0) \int_{\mathbb{R}} p_{t+\delta-s}(z) - p_{t-s}(z) dz ds \right)^{k/2} \\ &\leq C \left(\int_0^t s^{-\beta/\alpha} (\log(s+\delta) - \log(s)) ds \right)^{k/2}. \end{aligned}$$

By integrating by parts,

$$(2.25) \quad \begin{aligned} &\int_0^t s^{-\beta/\alpha} (\log(s+\delta) - \log(s)) ds \\ &= \frac{\alpha}{\alpha-\beta} \log(1+\delta/t) t^{(\alpha-\beta)/\alpha} + \frac{\alpha}{\alpha-\beta} \int_0^t s^{(\alpha-\beta)/\alpha} \frac{\delta}{s(s+\delta)} ds \\ &= I_3 + I_4. \end{aligned}$$

For I_4 ,

$$(2.26) \quad \begin{aligned} \frac{\alpha}{\alpha-\beta} \int_0^t s^{(\alpha-\beta)/\alpha} \frac{\delta}{s(s+\delta)} ds &= \left(\frac{\alpha}{\alpha-\beta} \right)^2 \int_0^{\frac{(\alpha-\beta)t}{\alpha}} \frac{\delta}{\mu^{\alpha/(\alpha-\beta)} + \delta} d\mu \\ &= \left(\frac{\alpha}{\alpha-\beta} \right)^2 \delta^{\frac{(\alpha-\beta)}{\alpha}} \int_0^{\frac{(\alpha-\beta)t}{\alpha} \delta^{\frac{(\beta-\alpha)}{\alpha}}} \frac{1}{1 + \nu^{\alpha/(\alpha-\beta)}} d\mu \\ &\leq \left(\frac{\alpha}{\alpha-\beta} \right)^2 \delta^{\frac{(\alpha-\beta)}{\alpha}} \int_0^\infty \frac{1}{1 + \nu^{\alpha/(\alpha-\beta)}} d\mu \\ &\leq C \delta^{\frac{(\alpha-\beta)}{\alpha}}. \end{aligned}$$

Next, we will prove that for any $\mu > 0$, $r \in [\frac{1}{2}, 1]$,

$$(2.27) \quad 0 < \log(1+\mu) \leq \mu^r.$$

For $r = 1$, it is obvious that $0 < \log(1+\mu) \leq \mu$. In the following, we only consider the case $r \in [\frac{1}{2}, 1)$. For $\mu > 0$, let $h(\mu) = \log(1+\mu) - \mu^r$. Then

$$(2.28) \quad h'(\mu) = \frac{\mu^{1-r} - (1+\mu)r}{(1+\mu)\mu^{1-r}}.$$

Let $l(\mu) = \mu^{1-r} - (1+\mu)r$. It is easy to get the maximum $l_{max} = l(\mu)|_{\mu=(\frac{1-r}{r})^{\frac{1}{r}}} = \frac{r^2}{1-r} \left[\left(\frac{1-r}{r} \right)^{\frac{1}{r}} - \frac{1-r}{r} \right]$. Since $r \in [\frac{1}{2}, 1)$, we get $l_{max} \leq 0$. By (2.28), $h'(\mu) \leq 0$. This together with $h(0) = 0$ yield that $h(\mu) \leq 0$. Therefore (2.27) holds. Since $\beta \in (0, \frac{\alpha}{2}]$, then $\frac{\alpha-\beta}{\alpha} \in [\frac{1}{2}, 1)$. By (2.27), we have

$$(2.29) \quad 0 < \log(1+\delta/t) \leq (\delta/t)^{(\alpha-\beta)/\alpha}.$$

Thus, for I_3 , we get

$$(2.30) \quad I_3 \leq C \delta^{\frac{(\alpha-\beta)}{\alpha}}.$$

By (2.24), (2.25), (2.26) and (2.30), we have

$$(2.31) \quad E(|I_{\beta,t+\delta}(x) - I_{\beta,t}(x)|^k) \leq C \delta^{\frac{(\alpha-\beta)k}{2\alpha}}.$$

□

2.4. Spatial and temporal Hölder continuity.

Theorem 2.6. *For all $k \geq 2$, $\alpha \in (1, 2]$, $\beta \in (0, \frac{\alpha}{2}]$, $\rho \in (0, 1)$, $b \in (0, 1 - \frac{1}{\alpha})$ and $x, y \in \mathbb{R}$,*

$$(2.32) \quad E(|u_{\beta,t}(x) - u_{\beta,s}(y)|)^k \leq C(|x - y|^{kc} + |t - s|^{kd}),$$

where C_1, C_2 are positive constants, $c \in (0, \frac{\alpha b}{2} \wedge \rho)$, $d \in (0, \frac{(\alpha - \beta)}{2\alpha} \wedge \frac{\rho}{\alpha})$.

Proof. By Lemma 2.3, Theorem 2.4 and Theorem 2.5, we can draw the conclusion. □

3. ACKNOWLEDGEMENTS

This work is partial supported by National Natural Science Foundation of China under the contract No.11571269, China Postdoctoral Science Foundation Funded Project under contracts No.2015M572539 and No.2016T90899 and Shaanxi Province Postdoctoral Science Foundation Funded Project.

REFERENCES

- [1] P. Bezdek, On weak convergence of stochastic heat equation with colored noise, Stoch. Proc. Appl. 126 (2016) 2860-2875.
- [2] J. L. Vázquez, A. de Pablo, F. Quirós, A. Rodriguez, Classical solutions and higher regularity for nonlinear fractional diffusion equations. To appear in J. Eur. Math. Soc. (JEMS). arXiv:1311.7427.
- [3] R. M. Balan, D. Conus, Intermittency for the wave and heat equations with fractional noise in time, Ann. Probab. 44 (2016) 1488-1534.
- [4] R. M. Balan, C.A. Tudor, Stochastic heat equation with multiplicative fractional-colored noise. J. Theor. Prob. 23 (2010) 834-870.
- [5] Z.-Q. Chen, K. Panki, R. Song, Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation, Ann. Probab. 40 (2012) 2483-2358.
- [6] T. Jakubowski, G. Serafin, Stable estimates for source solution of critical fractal Burgers equation, Nonlinear. Anal. 130 (2016) 396-407.
- [7] K. Bogdan, T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, Commun. Math. Phys. 271 (2007) 179-198.
- [8] D. Conus, M. Joseph, D. Khoshnevisan and S.-Y. Shin, On the chaotic character of the stochastic heat equation, II, Probab. Theory Related Fields. 156 (2013) 483-533.
- [9] R. Dalang, Extending the martingale measure stochastic integral with applications to spatially homogeneous S.P.D.E.'s, Electron. J. Probab. 4 (1999) 1-29.
- [10] D. Khoshnevisan, Analysis of Stochastic Partial Differential Equations, vol 119. American Mathematical Soc., 2014.
- [11] M. Foondun, D. Khoshnevisan, On the stochastic heat equation with spatially-colored random forcing, Trans. Amer. Math. Soc. 365 (2013) 409-458.
- [12] M. Ferrante, M. Sanz-Solé, SPDEs with coloured noise: analytic and stochastic approaches, ESAIM Prob. Stat. 10 (2006) 380-405.
- [13] R.C. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, Y. Xiao, A Minicourse on Stochastic Partial Differential Equations. Salt Lake City, Utah, 2006.
- [14] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, New Jersey, 1970.

SCHOOL OF MATHEMATICS AND STATISTICS, XI'AN JIAOTONG UNIVERSITY, XI'AN 710049, CHINA
E-mail address: kexueli@gmail.com